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PROCESSOR:

COMMENTS ON J. VON NEUMANN'S
"THE PROBLEM OF OPTIMAL ASSIGNMENT IN
A TWO-PERSON GAME"

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SUMMARY

Certain arguments in J. von Neumann's paper reducing the optimal assignment problem to a two-person game can be simplified. The principal contribution here appears in section (e) where a simple observation produces a proof that all vertices of the convex of solutions (of a related continuous problem) are permutations; hence, admissible solutions to the original combinatorial problem. (This is a modification of the author's proof that optimal solution to the transportation problem is integral if the row and column totals are integers.) Corresponding proofs by J. von Neumann and an earlier one by G. Birkhoff are comparatively more involved. The present proof depends on a well-known and easily verified theorem that the vertex of a convex defined by m-linear equations in n-non-negative variables (considered as a point in R_n) has at most m positive components.

COMMENTS ON J. VON NEUMANN'S "THE PROBLEM OF OPTIMAL ASSIGNMENT IN A TWO-PERSON GAME"

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J. von Neumann in his paper [1]* shows that the problem of optimal assignment of m men to m jobs (where the payoff is b_{ij} if the i-th man is assigned to the j-th job) is equivalent to the following game: Player 1 chooses a square in a checker board while simultaneously Player 2 chooses either a row or a column; if the square chosen by Player 1 is in the row or column chosen by Player 2, the payoff to Player 2 is $a_{ij} = 1/b_{ij}$. To establish equivalence:

(a) Choose $a_{ij} > 0$ by replacing original a_{ij} by $a_{ij} + u_i + v_j$ if necessary.

(b) Player 1's mixed strategy is to choose $x_{ij} \geq 0$ so that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = 1$$

$$\sum_{i=1}^{m} x_{ij} a_{ij} \leq M \qquad (j=1,2,\cdots,n)$$

$$\sum_{j=1}^{n} x_{ij} a_{ij} \leq M \qquad (i=1,2,\cdots,m)$$

$$M = \text{Min.}$$

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* See References at end of paper.

(c) Set

$$\frac{x_{ij}a_{ij}}{M} = z_{ij};$$

then the above game is equivalent to the linear problem.

$$\sum_i z_{ij} \leq 1 \qquad z_{ij} \geq 0 \qquad (j=1,2,\cdots,n)$$

$$\sum_j z_{ij} \leq 1 \qquad (i=1,2,\cdots,m)$$

$$\sum_i \sum_j \frac{z_{ij}}{a_{ij}} = \frac{1}{M} = \text{Max}$$

since $a_{ij} > 0$, $M > 0$.

(d) To show optimal solution of the above implies $\sum_i z_{ij} = 1$ for all i and $\sum_j z_{ij} = 1$ for all j, assume on the contrary an optimal solution z_{ij} satisfying

$$\sum_j z_{ij} = 1 - r_i \qquad (i=1,2,\cdots,m)$$

$$\sum_i z_{ij} = 1 - c_j \qquad (j=1,2,\cdots,n)$$

$$\sum \frac{z_{ij}}{a_{ij}} = M_0.$$

Consider now the transportation problem

$$\sum_j z'_{ij} = r_i \qquad (i=1,2,\cdots,m)$$

$$\sum_i z'_{ij} = c_j \qquad (j=1,2,\cdots,n)$$

$$\sum \frac{z'_{ij}}{a_{ij}} = M_1 .$$

Arbitrary solutions $z'_{ij} \geq 0$ to any transportation problem

$$\sum r_i = \sum c_j = K > 0$$

are easily constructed; the simplest is to set

$$z'_{ij} = \frac{1}{K} r_i \cdot c_j$$

where not all $z'_{ij} = 0$ since $\sum_i \sum_j z'_{ij} = K$. It is clear then that $M_1 > 0$ and

$$\sum_i (z_{ij} + z'_{ij}) = 1 \qquad (j=1,2,\cdots,n)$$

$$\sum_j (z_{ij} + z'_{ij}) = 1 \qquad (i=1,2,\cdots,m)$$

$$\sum_i \sum_j \frac{z_{ij} + z'_{ij})}{a_{ij}} = M_0 + M_1 ,$$

contradicting optimality of the solution z_{ij}.

(e) To show that extreme points of the convex of solutions of

$$\sum_{i=1}^{n} z_{ij} = 1 \qquad (j=1,2,\cdots,n)$$

$$\sum_{j=1}^{n} z_{ij} = 1 \qquad (i=1,2,\cdots,m)$$

is a permutation, it is only necessary to note that at an extreme point no more than $2n - 1$ of the $z_{ij} > 0$ because the number of independent equations are $2n - 1$. At least one row must have exactly one $z_{ij} > 0$ because if each had two or more $z_{ij} > 0$ then there would be at least $2n$ nonzero z_{ij} at an extreme point, a contradiction. The value of this nonzero $z_{ij} = 1$ (i.e., equal to the row total). Eliminating the row and column of this z_{ij}, the argument can be repeated with the reduced array until there are no more rows and columns left—each time one nonzero z_{ij} in the solution is evaluated as $z_{ij} = 1$. It follows, since all other z_{ij} are zero, that the vertex is a permutation.

(f) D. R. Fulkerson in an unpublished paper has considered the multi-index transportation problem and has shown that there are extreme points which need not be permutations.

REFERENCES

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2. George B. Dantzig, "Application of Simplex Method to a Transportation Problem," Activity Analysis of Production and Allocation, Koopmans (Ed.), 1951.

3. ——————. Class notes for course in Linear Programming, American University (Fall 1951).